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Rearrangement-Function Inequalities and Interpolation Theory

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Using the tools of Real Interpolation Theory, we develop a general method for proving rearrangement-function inequalities for important classes of operators.

Key Words: Real Interpolation; weak type; rearrangements; *BMO*.

1. INTRODUCTION

There are numerous rearrangement-function inequalities in the literature, connecting the rearrangements of functions, the rearrangements of their sharp-functions, and the rearrangements of their maximal functions. These inequalities arise naturally as consequences of good-λ inequalities, see for example [7], see also [1, 2, 10, 11]. Let us recall one of the simplest such inequalities.

It is well known that for all $t>0$ the Hardy–Littlewood maximal function, Mf, satisfies,

$$(Mf)^*(t) \leqslant \frac{C}{t}\int_0^t f^*.$$

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(Here and in the rest of the paper C stands for a generic constant). More generally, one can show that if T is a quasi-linear operator satisfying

$$||Tf||_{1,\infty} \leqslant C||f||_1 \tag{1.1}$$

and

$$||Tf||_{\infty} \leqslant C||f||_{\infty} \tag{1.2}$$

then

$$(Tf)^*(t) \leqslant \frac{C}{t}\int_0^t f^*. \tag{1.3}$$

Since

$$\sup_{0<s<t} sf^*(s) \backsim K(t, f; L(1,\infty), L^\infty)$$

and

$$\int_0^t f^* = K(t, f; L^1, L^\infty)$$

we see that (1.3) is equivalent to

$$K(t, Tf; L(1,\infty), L^\infty) \leqslant CK(t, f; L^1, L^\infty). \tag{1.4}$$

In retrospect, it is clear from the definition of the K-functional that (1.1) and (1.2) imply (1.4). Thus the rearrangement-function inequality, (1.3), follows from calculations of K-functionals.

Let us consider a second example. For the Hilbert transform, H, it follows from results to be presented in Section 5 that, for every $\gamma > 1$,

$$(Hf)^*(t) - (Hf)^*(\gamma t) \leqslant C(f^\#)^*\left(\frac{t}{C}\right), \tag{1.5}$$

where $f^\#$ is the Fefferman–Stein sharp function.

This inequality follows from the continuity properties of H:

$$H : L^1 \mapsto L(1,\infty)$$

and

$$H : BMO \mapsto BMO$$

which imply an inequality between $K(t, Hf; L(1,\infty), BMO)$ and $K(t, f; L^1, BMO)$. This inequality together with calculations of K and E-functionals with respect to weak-type classes near the endpoints of interpolation scales implies (1.5), and so proves the inequality for all operators with the same continuity properties.

Weak-type classes were defined in [14, 15]. In some cases the weak-type classes contain well-known spaces, such as BMO, which are natural range spaces for classical operators. We will see that calculations with the K and the E-functionals for interpolation couples which include the weak-type classes, when applied to the $L(p,q)$ scale, imply rearrangement-function inequalities, such as (1.5), for important classes of operators. These inequalities, which up to now were proved ad hoc, thus become part of Real Interpolation Theory.

2. CALCULATION OF THE K-FUNCTIONAL FOR SOME WEAK-TYPE CLASSES

Let (A_0, A_1) be an interpolation couple. Here A_j are quasi-Banach groups, that is to say, there exist functions $||\cdot||_{A_j} : A_j \mapsto \mathbb{R}_+$ so that $||a||_{A_j} = 0 \Leftrightarrow a = 0$, $||a||_{A_j} = ||-a||_{A_j}$ and $||\cdot||_{A_j}$ satisfy the quasi-triangle inequality:

$$||a+b||_{A_j} \leqslant c_j(||a||_{A_j} + ||b||_{A_j}).$$

Let

$$K_r(t, a; A_0, A_1) = \inf\{(||a_0||^r_{A_0} + t^r||a_1||^r_{A_1})^{\frac{1}{r}} : a_0 + a_1 = a, a_j \in A_j\}.$$

We denote $K_1 = K$. In [15] we defined weak-type classes and showed their usefulness in Real Interpolation Theory. It turns out that a somewhat more general definition of these classes is more useful.

Let $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$. We denote the least concave majorant of ϕ by $\hat{\phi}$.

Definition 2.1. Let (A_0, A_1) be an interpolation couple. Let

$$g : \mathbb{R}_+ \times (A_0 + A_1) \mapsto \mathbb{R}_+$$

be such that

$$\hat{g} \backsim K(\cdot\,, \cdot\,; A_0, A_1). \tag{2.1}$$

For $0 \leqslant \varepsilon < \infty$, $0 < r < \infty$, and $1 < \gamma < \infty$, we denote

$$||a||_{W_K[A_0, A_1; \varepsilon, \gamma, r, g]} = \sup_{t>0} \; [g^r(\gamma t, a) - \varepsilon^r g^r(t, a)]_+^{\frac{1}{r}} \tag{2.2}$$

and

$$W_K[A_0, A_1; \varepsilon, \gamma, r, g] = \{a \in A_0 + A_1 : ||a||_{W_K[A_0,A_1;\varepsilon,\gamma,r,g]} < \infty\}.$$

If a function, g, appears as a parameter in $W_K[A_0, A_1; \varepsilon, \gamma, r, g]$ we assume implicitly that it satisfies (2.1).

In [15] there is a similar definition of classes $W_K(A_0, A_1; \varepsilon, \gamma, r, g)$. The only difference is that in classes $W_K(A_0, A_1; \varepsilon, \gamma, r, g)$ it is assumed that

$$g \backsim K(\cdot\,,\cdot\,; A_0, A_1). \tag{2.3}$$

Of course, (2.3) implies (2.1) so that we will write $W_K(A_0, A_1; \varepsilon, \gamma, r, g)$ if (2.3) holds. Also we denote $W_K(A_0, A_1; \varepsilon, \gamma, r, K)$ by $W_K(A_0, A_1; \varepsilon, \gamma, r)$.

In [15] we showed that the choice of r and g does not affect the interpolation result, and one might be tempted to eliminate these parameters. However, since $||\cdot||_{W_K[A_0,A_1;\varepsilon,\gamma,r,g]}$ is defined by a difference, replacing g by a function, g_1, satisfying

$$g \backsim g_1$$

may change the class W_K. In most cases K is known only up to equivalence and we define g to be the expression calculated to be equivalent to the K-functional.

The set $W_K := W_K[A_0, A_1; \varepsilon, \gamma, r, g]$ is, in general, not a group. However there are cases where important spaces are embedded in W_K, see [15]. For this reason we consider

$$K(t, a; W_K, A_1) := \inf\{||a_0||_{W_K} + t||a_1||_{A_1} : a_0 + a_1 = a, a_0 \in W_K, a_1 \in A_1\}.$$

Thus if S is a space and $||\cdot||_{W_K} \leqslant C||\cdot||_S$ and $a \in A_0 + A_1$ we have

$$K(t, a; W_K, A_1) \leqslant CK(t, a; S, A_1).$$

The interpolation theorem proved in [15] allows one to derive in a systematic way interpolation theorems for the embedded spaces.

In this paper we take the next step: we relate $K(t, a; W_K[A_0, A_1; 1, \gamma, r, g], A_1)$ to the parameter function g and so, indirectly, to $K(t, a; A_0, A_1)$. As will become clear in the applications this will enable us, in the context of $L(p, q)$ spaces, to get rearrangement-function inequalities as corollaries of continuity properties of operators on the endpoints of the interpolation scales. These inequalities imply, of course, the continuity properties on the intermediate spaces.

For the rest of the paper we will consider only the case $\varepsilon = 1$ in (2.2).

DEFINITION 2.2.

$$g^{\wedge}(t,a) = g^{\wedge}(t,a,\gamma,r;A_0,A_1) = \sup_{s\leqslant t}\,[g^r(\gamma s,a;A_0,A_1) - g^r(s,a;A_0,A_1)]_+^{\frac{1}{r}}.$$

Observe that $g^{\wedge}$ is the least non-decreasing majorant of the function which defines the class $W_K[A_0,A_1;1,\gamma,r,g]$. Thus

$$\lim_{t\to\infty} g^{\wedge}(t,a) = \sup_{s>0}\,[g^r(\gamma s,a;A_0,A_1) - g^r(s,a;A_0,A_1)]_+^{\frac{1}{r}}$$
$$= ||a||_{W_K[A_0,A_1;1,\gamma,r,g]}.$$

It will turn out that $g^{\wedge}$ is equivalent to a K-functional which is, of course, a non-decreasing function. Thus it is natural to take a least non-decreasing majorant.

If (A_0,A_1) is an interpolation couple of Banach spaces then it is easy to see that

$$K(t,a+a';A_0,A_1)\leqslant K(t,a;A_0,A_1) + t||a'||_{A_1}.$$

Since we are considering functions, g, which satisfy only (2.1), and since we are considering more general interpolation couples, we have to postulate a similar condition in the following theorem.

THEOREM 2.3. *Assume that g satisfies* (2.1) *and that there exist an integer* $m\geqslant 0$, *and a constant* $\lambda\geqslant 0$ *so that for all* $a\in A_0+A_1$, $a'\in A_1$ *and all* $s>0$, *we have*

$$g^r(s,a+a')\leqslant g^r(\gamma^m s,a) + \lambda s^r||a'||_{A_1}^r. \tag{2.4}$$

Then for every $a\in A_0+A_1$ *and* $t>0$ *we have*

$$g^{\wedge}(t,a)\leqslant cK(t,a;W_K[A_0,A_1;1,\gamma,r,g],A_1), \tag{2.5}$$

where $c = 4^{\frac{1}{r}}\max\left\{(m+1)^{\frac{1}{r}};\lambda^{\frac{1}{r}}\gamma\right\}$.

Proof. We denote $\tilde{A}_0 = W_K[A_0,A_1;1,\gamma,r,g]$. Since K is a non-decreasing function, it suffices to prove

$$[g^r(\gamma t,a) - g^r(t,a)]^{\frac{1}{r}}\leqslant cK(t,a;\tilde{A}_0,A_1).$$

Let $\eta>0$ be given and let $b\in\tilde{A}_0$ be such that

$$||b||_{\tilde{A}_0}^r + t^r||a-b||_{A_1}^r\leqslant K_r^r(t,a;\tilde{A}_0,A_1)+\eta.$$

Then

$$\begin{aligned} g^r(\gamma t, a) - g^r(t, a) &= [g^r(\gamma t, a) - g^r(\gamma t, b)] + [g^r(\gamma t, b) - g^r(t, b)] \\ &\quad + [g^r(t, b) - g^r(t, a)] \\ &\leqslant [g^r(\gamma t, a) - g^r(\gamma t, b)] + ||b||^r_{\tilde{A}_0} + [g^r(t, b) - g^r(t, a)]. \end{aligned}$$

But, by (2.4),

$$\begin{aligned} g^r(\gamma t, a) - g^r(\gamma t, b) &\leqslant g^r(\gamma^{m+1} t, b) + \lambda \gamma^r t^r ||a - b||^r_{A_1} - g^r(\gamma t, b) \\ &= \lambda \gamma^r t^r ||a - b||^r_{A_1} + \sum_{j=0}^{m-1} [g^r(\gamma^{m+1-j} t, b) - g^r(\gamma^{m-j} t, b)] \\ &\leqslant \lambda \gamma^r t^r ||a - b||^r_{A_1} + m||b||^r_{\tilde{A}_0} \end{aligned}$$

and similarly

$$\begin{aligned} g^r(t, b) - g^r(t, a) &= [g^r(t, b) - g^r(\gamma^{-m} t, b)] + [g^r(\gamma^{-m} t, b) - g^r(t, a)] \\ &\leqslant m||b||^r_{\tilde{A}_0} + \lambda \gamma^{-mr} t^r ||b - a||^r_{A_1}. \end{aligned}$$

Thus

$$\begin{aligned} g^r(\gamma t, a) - g^r(t, a) &\leqslant (2m + 1)||b||^r_{\tilde{A}_0} + \lambda(\gamma^r + \gamma^{-mr}) t^r ||a - b||^r_{A_1} \\ &\leqslant 2 \max\{m + 1; \lambda \gamma^r\}(||b||^r_{\tilde{A}_0} + t^r ||a - b||^r_{A_1}) \\ &\leqslant 2 \max\{m + 1; \lambda \gamma^r\}(K^r_r(t, a; \tilde{A}_0, A_1) + \eta) \end{aligned}$$

and so

$$\begin{aligned} (g^r(\gamma t, a) - g^r(t, a))^{\frac{1}{r}} &\leqslant 2^{\frac{1}{r}} \max\{(m + 1)^{\frac{1}{r}}; \lambda^{\frac{1}{r}} \gamma\} K_r(t, a; \tilde{A}_0, A_1) \\ &\leqslant 4^{\frac{1}{r}} \max\{(m + 1)^{\frac{1}{r}}; \lambda^{\frac{1}{r}} \gamma\} K(t, a; \tilde{A}_0, A_1). \quad \blacksquare \end{aligned}$$

Remark 2.4. Given a function, g, the application of Theorem 2.3 requires verification of $\hat{g} \backsim K(\cdot\,, \cdot\,; A_0, A_1)$. The calculation of $\hat{g}$ is, in general, not easy. The following result simplifies the problem in some cases.

We have already mentioned the least non-decreasing majorant of a function on $\mathbb{R}_+$; let us introduce an appropriate notation. Given $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ we denote $\overset{\nearrow}{g}(t) = \sup_{s \leqslant t} g(s)$.

We state that if $\overset{\nearrow}{g} \backsim K(\cdot\,,\cdot\,; A_0, A_1)$ then $\hat{g} \backsim K(\cdot\,,\cdot\,; A_0, A_1)$.

For the proof denote $K(\cdot\,,\cdot\,; A_0, A_1)$ by K. Clearly $\hat{K} = K$. Since $g \leqslant \overset{\nearrow}{g}$ and $\overset{\nearrow}{g} \backsim K$, we have $\hat{g} \leqslant (\overset{\nearrow}{g})^{\frown} \leqslant C\hat{K} = CK$. Conversely, since $\hat{g}$ is a non-negative concave function on $\mathbb{R}_+$, it is a non-decreasing function. Therefore $\overset{\nearrow}{g} \leqslant \hat{g}$ and so $K \leqslant C\overset{\nearrow}{g} \leqslant C\hat{g}$.

The next theorem shows that Theorem 2.3 is, in a sense, best possible. Taking $g = K_r(\cdot\,,\cdot\,; A_0, A_1)$, inequality (2.5) becomes an equivalence.

Theorem 2.5. *For every $a \in A_0 + A_1$ and $t > 0$ we have*

$$\frac{1}{C(c_1, r, \gamma)} K_r^{\wedge}(t, a) \leqslant K(t, a; W_K(A_0, A_1; 1, \gamma, r, K_r), A_1) \leqslant C(c_1, r, \gamma) K_r^{\wedge}(t, a),$$

where c_1 is the constant in the quasi-triangle inequality for A_1.

Proof. We denote $\tilde{A}_0 = W_K(A_0, A_1; 1, \gamma, r, K_r)$.

Let us show that (2.4) holds for K_r with $\lambda = c_1^r 2^{(r-1)_+}$ and m so that $\gamma^{mr} \geqslant \lambda$, i.e., for all $a \in A_0 + A_1$, $a' \in A_1$ we have

$$K_r^r(s, a + a'; A_0, A_1) \leqslant K_r^r(\gamma^m s, a; A_0, A_1) + \lambda s^r ||a'||_{A_1}^r. \tag{2.6}$$

Let $\eta > 0$ be given. Let $a = a_0 + a_1$ where $a_j \in A_j$ are such that

$$||a_0||_{A_0}^r + \gamma^{mr} s^r ||a_1||_{A_1}^r \leqslant K_r^r(\gamma^m s, a; A_0, A_1) + \eta.$$

Then

$$\begin{aligned} K_r^r(s, a + a'; A_0, A_1) &= K_r^r(s, a_0 + (a_1 + a'); A_0, A_1) \\ &\leqslant ||a_0||_{A_0}^r + s^r ||a' + a_1||_{A_1}^r \\ &\leqslant ||a_0||_{A_0}^r + (2^{(r-1)_+}) c_1^r s^r ||a_1||_{A_1}^r + (2^{(r-1)_+}) c_1^r s^r ||a'||_{A_1}^r \\ &\leqslant K_r^r(\gamma^m s, a; A_0, A_1) + \lambda s^r ||a'||_{A_1}^r + \eta, \end{aligned}$$

which proves (2.6). From Theorem 2.3 it follows

$$K_r^{\wedge}(t, a) \leqslant C(c_1, r, \gamma) K(t, a; \tilde{A}_0, A_1).$$

Let us show that

$$K_r(t, a; \tilde{A}_0, A_1) \leqslant C(c_1, r, \gamma) K_r^{\wedge}(t, a).$$

For $\eta > 0$ we choose $b \in A_0$ so that

$$||b||_{A_0}^r + \gamma^r t^r ||a - b||_{A_1}^r \leqslant K_r^r(\gamma t, a; A_0, A_1) + \eta. \tag{2.7}$$

We have

$$\begin{aligned} &||b||_{A_0}^r + \gamma^r t^r ||a - b||_{A_1}^r - \eta \\ &\quad \leqslant K_r^r(\gamma t, a; A_0, A_1) \\ &\quad = [K_r^r(\gamma t, a; A_0, A_1) - K_r^r(t, a; A_0, A_1)] + K_r^r(t, a; A_0, A_1) \\ &\quad \leqslant [K_r^\wedge(t, a)]^r + ||b||_{A_0}^r + t^r ||a - b||_{A_1}^r, \end{aligned}$$

which gives us

$$(\gamma^r - 1) t^r ||a - b||_{A_1}^r \leqslant [K_r^\wedge(t, a)]^r + \eta. \tag{2.8}$$

Let us estimate

$$||b||_{\tilde{A}_0} = \sup_{s>0} \, (K_r^r(\gamma s, b; A_0, A_1) - K_r^r(s, b; A_0, A_1))^{\frac{1}{r}}.$$

If $0 < s \leqslant t$ then, by (2.6),

$$\begin{aligned} &K_r^r(\gamma s, b; A_0, A_1) - K_r^r(s, b; A_0, A_1) \\ &\quad \leqslant K_r^r(\gamma^{m+1} s, a; A_0, A_1) + \lambda \gamma^r s^r ||a - b||_{A_1}^r - K_r^r(s, b; A_0, A_1) \\ &\quad \leqslant K_r^r(\gamma^{m+1} s, a; A_0, A_1) - K_r^r(\gamma^{-m} s, a; A_0, A_1) \\ &\qquad + K_r^r(\gamma^{-m} s, a; A_0, A_1) - K_r^r(s, b; A_0, A_1) + \lambda \gamma^r s^r ||a - b||_{A_1}^r. \end{aligned}$$

But

$$K_r^r(\gamma^{-m} s, a; A_0, A_1) - K_r^r(s, b; A_0, A_1) \leqslant \lambda \gamma^{-mr} s^r ||a - b||_{A_1}^r \leqslant s^r ||a - b||_{A_1}^r$$

and

$$\begin{aligned} &K_r^r(\gamma^{m+1} s, a; A_0, A_1) - K_r^r(\gamma^{-m} s, a; A_0, A_1) \\ &\quad = \sum_{j=0}^{2m} (K_r^r(\gamma^{m+1-j} s, a; A_0, A_1) - K_r^r(\gamma^{m-j} s, a; A_0, A_1)) \\ &\quad \leqslant (2m+1)[K_r^\wedge(\gamma^m t, a; A_0, A_1)]^r. \end{aligned}$$

Hence

$$\sup_{0<s\leqslant t}[K_r^r(\gamma s,b;A_0,A_1)-K_r^r(s,b;A_0,A_1)]$$
$$\leqslant(2m+1)[K_r^\wedge(\gamma^m t,a)]^r+(1+\lambda\gamma^r)t^r||a-b||_{A_1}^r$$

so that by (2.8),

$$\sup_{0<s\leqslant t}[K_r^r(\gamma s,b;A_0,A_1)-K_r^r(s,b;A_0,A_1)]$$
$$\leqslant(2m+1)[K_r^\wedge(\gamma^m t,a)]^r+\frac{(1+\lambda\gamma^r)}{\gamma^r-1}([K_r^\wedge(t,a)]^r+\eta).$$

We have shown that for $0\leqslant s\leqslant t$

$$\sup_{0<s\leqslant t}[K_r^r(\gamma s,b;A_0,A_1)-K_r^r(s,b;A_0,A_1)]^{\frac{1}{r}}\leqslant C(c_1,r,\gamma)(K_r^\wedge(\gamma^m t,a)+\eta).$$

Let us consider the case $s>t$:

$$\begin{aligned}
&K_r^r(\gamma s,b;A_0,A_1)-K_r^r(s,b;A_0,A_1)\\
&\quad\leqslant||b||_{A_0}^r-K_r^r(t,b;A_0,A_1)\\
&\quad=||b||_{A_0}^r-K_r^r(\gamma t,b;A_0,A_1)+K_r^r(\gamma t,b;A_0,A_1)-K_r^r(t,b;A_0,A_1)\\
&\quad\leqslant||b||_{A_0}^r-K_r^r(\gamma t,b;A_0,A_1)\\
&\qquad+\sup_{0<s\leqslant t}[K_r^r(\gamma s,b;A_0,A_1)-K_r^r(s,b;A_0,A_1)]^{\frac{1}{r}}\\
&\quad\leqslant||b||_{A_0}^r-K_r^r(\gamma t,b;A_0,A_1)+C(c_1,r,\gamma)[K_r^\wedge(\gamma^m t,a)+\eta]^r.
\end{aligned}$$

From (2.7) it follows that

$$||b||_{A_0}^r\leqslant K_r^r(\gamma t,a;A_0,A_1)+\eta$$

so that

$$\begin{aligned}
&K_r^r(\gamma s,b;A_0,A_1)-K_r^r(s,b;A_0,A_1)\\
&\quad\leqslant K_r^r(\gamma t,a;A_0,A_1)-K_r^r(\gamma t,b;A_0,A_1)\\
&\qquad+C(c_1,r,\gamma)[K_r^\wedge(\gamma^m t,a)+\eta]^r+\eta\\
&\quad=K_r^r(\gamma t,a;A_0,A_1)-K_r^r(\gamma^{1-m}t,a;A_0,A_1)\\
&\qquad+K_r^r(\gamma^{1-m}t,a;A_0,A_1)-K_r^r(\gamma t,b;A_0,A_1)\\
&\qquad+C(c_1,r,\gamma)[K_r^\wedge(\gamma^m t,a)+\eta]^r+\eta.
\end{aligned}$$

As before

$$K_r^r(\gamma t, a; A_0, A_1) - K_r^r(\gamma^{1-m} t, a; A_0, A_1) \leqslant m[K_r^\wedge(t, a)]^r.$$

Also, by (2.6),

$$\begin{aligned} &K_r^r(\gamma^{1-m} t, a; A_0, A_1) - K_r^r(\gamma t, b; A_0, A_1) \\ &\quad \leqslant \gamma^{mr}(\gamma^{1-m} t)^r ||a-b||_{A_1}^r \\ &\quad = \gamma^r t^r ||a-b||_{A_1}^r \end{aligned}$$

and by (2.8)

$$\gamma^r t^r ||a-b||_{A_1}^r \leqslant \frac{\gamma^r}{\gamma^r - 1}([K_r^\wedge(t, a)]^r + \eta)$$

so that

$$\begin{aligned} &K_r^r(\gamma t, a; A_0, A_1) - K_r^r(\gamma t, b; A_0, A_1) \\ &\quad \leqslant m[K_r^\wedge(t, a)]^r + \frac{\gamma^r}{\gamma^r - 1}([K_r^\wedge(t, a)]^r + \eta) \\ &\quad \leqslant C(c_1, r, \gamma)([K_r^\wedge(t, a)]^r + \eta) \\ &\quad \leqslant C(c_1, r, \gamma)([K_r^\wedge(\gamma^m t, a)]^r + \eta). \end{aligned}$$

Thus

$$\begin{aligned} &K_r^r(\gamma s, b; A_0, A_1) - K_r^r(s, b; A_0, A_1) \\ &\quad \leqslant C(c_1, r, \gamma)([K_r^\wedge(\gamma^m t, a; A_0, A_1)]^r + \eta) + \eta. \end{aligned}$$

This proves

$$||b||_{\tilde{A}_0}^r \leqslant C(c_1, r, \gamma)[K_r^\wedge(\gamma^m t, a; A_0, A_1)]^r + \eta.$$

By (2.8) we have

$$\begin{aligned} &K_r^r(t, a; \tilde{A}_0, A_1) \\ &\quad \leqslant ||b||_{\tilde{A}_0}^r + t^r ||a-b||_{A_1}^r \\ &\quad \leqslant C(c_1, r, \gamma)[K_r^\wedge(\gamma^m t, a)]^r + \eta + \frac{1}{\gamma^r - 1}([K_r^\wedge(t, a)]^r + \eta) \end{aligned}$$

so that

$$K_r(t, a; \tilde{A}_0, A_1) \leqslant C(c_1, r, \gamma) K_r^\wedge(\gamma^m t, a).$$

Therefore

$$K_r(t,a;\tilde{A}_0,A_1)\leqslant\gamma^m K_r(t\gamma^{-m},a;\tilde{A}_0,A_1)\leqslant C(c_1,r,\gamma)K_r^{\wedge}(t,a). \quad \blacksquare$$

COROLLARY 2.6. *We denote*

$$\tilde{A}_1 = W_K(A_1,A_0;1,\gamma,r,K_r).$$

Then

$$K_r(t,a;A_0,\tilde{A}_1)\backsim t\sup_{t\leqslant u}\left(\frac{K^r(u,a;A_0,A_1)}{u^r}-\frac{K^r(\gamma u,a;A_0,A_1)}{(\gamma u)^r}\right)^{\frac{1}{r}}.$$

The following theorem connects Theorem 2.3 to concrete applications.

DEFINITION 2.7. Let T be a mapping to measurable functions. We say that T is subadditive if for all a,b in the domain of T both $T(a+b)\leqslant Ta+Tb$ and $|Ta|=|T(-a)|$ hold.

THEOREM 2.8. *Let (A_0,A_1) and (B_0,B_1) be two interpolation couples. Assume that B_0 and B_1 are spaces of measurable functions and that B_1 is a Banach lattice. Let $T:A_0+A_1\to B_0+B_1$ be a subadditive operator which satisfies*

$$||Ta||_{W_K[B_0,B_1;1,\gamma,r,g]}\leqslant M_0||a||_{A_0},$$

$$||Ta||_{B_1}\leqslant M_1||a||_{A_1}.$$

Assume also that g satisfies condition (2.4).

Then for all $t>0,\ \gamma>1,\ r>0,$ and $a\in A_0+A_1$

$$\hat{\boldsymbol{g}}(t,Ta,\gamma,r;B_0,B_1)\leqslant CM_0K\left(\frac{M_1}{M_0}t,a;A_0,A_1\right),$$

where $C=C(\gamma,r)$.

Proof. Let us denote in this proof

$$W_K = W_K[B_0,B_1;1,\gamma,r,g].$$

Let $a\in A_0+A_1$ and $t,\varepsilon>0$ be given. Let $a_0\in A_0$ and $a_1\in A_1$ be such that $a_0+a_1=a$ and

$$||a_0||_{A_0}+t||a_1||_{A_1}\leqslant K(t,a;A_0,A_1)+\varepsilon.$$

Then

$$\begin{aligned} K(t, Ta; W_K, B_1) &= K(t, Ta - Ta_0 + Ta_0; W_K, B_1) \\ &\leqslant ||Ta_0||_{W_K} + t||Ta - Ta_0||_{B_1}. \end{aligned}$$

But, since T is subadditive,

$$|Ta - Ta_0| \leqslant |Ta_1|$$

and since B_1 is a Banach lattice, we have

$$||Ta - Ta_0||_{B_1} \leqslant ||Ta_1||_{B_1}$$

so that

$$\begin{aligned} K(t, Ta; W_K, B_1) &\leqslant ||Ta_0||_{W_K} + t||Ta - Ta_0||_{B_1} \\ &\leqslant ||Ta_0||_{W_K} + t||Ta_1||_{B_1} \\ &\leqslant M_0||a_0||_{A_0} + M_1 t||a_1||_{A_1} \\ &\leqslant M_0\left(K\left(\frac{M_1}{M_0}t, a; A_0, A_1\right) + \varepsilon\right). \end{aligned}$$

Thus

$$K(t, Ta; W_K, B_1) \leqslant M_0 K\left(\frac{M_1}{M_0}t, a; A_0, A_1\right).$$

We apply Theorem 2.3 to estimate $K(t, Ta; W_K, B_1)$ from below. Since, by hypothesis, (2.4) holds,

$$g^{\wedge}(t, Ta) \leqslant CK(t, Ta; W_K, B_1) \leqslant CM_0 K\left(\frac{M_1}{M_0}t, a; A_0, A_1\right). \qquad \blacksquare$$

3. APPLICATIONS OF THE K-FUNCTIONAL

Let us consider first an inequality of Bagby and Kurtz [2]. If T is the maximal Calderón–Zygmund singular integral operator and $\mathscr{M}$ is the Hardy–Littlewood maximal function with respect to Lebesgue measure, and if ω is a weight function in the Muckenhoupt A_∞ class then

$$(Tf)^{*,\omega}(t) \leqslant C(\mathscr{M}f)^{*,\omega}\left(\frac{t}{2}\right) + (Tf)^{*,\omega}(2t), \tag{3.1}$$

where the superscript ω indicates rearrangement with respect to the measure $\omega(x)\,dx$.

As an illustration of the applicability of the methods of Section 2, for $\omega = 1$ we will prove a somewhat more general inequality.

Maximal Calderón–Zygmund operators map L^∞ to BMO and L^1 to $L(1,\infty)$ so that the following result generalizes (3.1).

Let T be a subadditive operator which satisfies:

$$T : L^1 \mapsto L(1,\infty),$$

$$T : L^\infty \mapsto BMO.$$

Then for all $\gamma > 1$

$$(Tf)^*(t) \leqslant C(T,\gamma)(\mathscr{M}f)^*(t) + (Tf)^*(\gamma t). \tag{3.2}$$

A related result was proved by Lerner [11].

With a stronger condition on the kernel of the Calderón–Zygmund operator the maximal operator maps BMO to BMO; this follows for example from Theorem 4.7 in [8]. For subadditive operators which satisfy

$$T : L^1 \mapsto L(1,\infty),$$

$$T : BMO \mapsto BMO,$$

we can prove

$$(Tf)^*(t) \leqslant C(T,\gamma)(f^{\#})^*(t) + (Tf)^*(\gamma t), \tag{3.3}$$

where $f^{\#}$ is the Fefferman–Stein maximal function. Both (3.2) and (3.3) can be proved directly using the results of [3], and the identification of the K-functionals for the pairs (L^1, L^∞), (L^1, BMO), and $(L(1,\infty), BMO)$.

We proceed with generalizations of these results. For example, instead of the space $L(1,\infty)$ in the range we can take $L(p,\infty)$ or L^p for any $0 < p < \infty$.

We recall some definitions and results.

In the sequel all cubes have sides parallel to the axes. The letter Q will stand for a cube. We say that a Borel measures on $\mathbb{R}^d, \mu$, satisfies the doubling condition if for all cubes, Q,

$$\mu(2Q) \leqslant c_\mu \mu(Q),$$

where we denote by $2Q$ the cube which has the same center as Q whose side length is twice that of Q. We call c_μ the doubling constant. We denote by μ_Q

the measure

$$\mu_Q(E) = \frac{\mu(Q \cap E)}{\mu(Q)}.$$

The Hardy–Littlewood maximal function with respect to the measure μ is defined as

$$\mathscr{M}_\mu f(x) = \sup_{Q \ni x} \int_Q |f| \, d\mu_Q. \tag{3.4}$$

We denote by $f^{*,\mu}$ the non-increasing rearrangement of f with respect to the measure μ, if it exists. We will denote

$$f^{**,\mu}(t) = \frac{1}{t} \int_0^t f^{*,\mu}(s) \, ds.$$

When μ is Lebesgue measure we write simply f^*, f^{**}.

In the sequel we continue to omit mention of the measure when we are referring to Lebesgue measure. Also when the underlying space is $\mathbb{R}^d$ we will omit mention of the space.

We denote

$$f_{Q,\mu} = \int_Q f \, d\mu_Q.$$

Given a cube Q_0 we denote

$$f^{\#,\mu}_{p,Q_0}(x) = \sup\left\{ \left(\int_Q |f - f_{Q,\mu}|^p \, d\mu_Q \right)^{\frac{1}{p}} : Q \subseteq Q_0 \text{ and } Q \ni x \right\}.$$

We also denote

$$f^{\#,\mu}_p(x) = \sup\left\{ \left(\int_Q |f - f_Q|^p \, d\mu_Q \right)^{\frac{1}{p}} : Q \ni x \right\}. \tag{3.5}$$

If $p = 1$ we write $f^{\#,\mu}$ for $f^{\#,\mu}_1$.

It is well known that if μ is a doubling measure then for all $p > 0$,

$$||f||_{BMO(\mu)} \backsim ||f^{\#,\mu}_p||_\infty.$$

THEOREM 3.1 (Bennett et al. [3]). *Let μ satisfy the doubling condition with doubling constant c_μ. Let $0 < \theta < \frac{1}{c_\mu^2}$ and $f \in L^1_{\mathrm{loc}}(\mu)$. Then for every cube, Q, and $0 < t < \frac{\theta}{2}\mu(Q)$ we have*

$$(fI_Q)^{**,\mu_Q}(t) - (fI_Q)^{*,\mu_Q}(t) \leqslant C(\mu, d)((fI_Q)^{\#,\mu_Q})^{*,\mu_Q}(t). \tag{3.6}$$

The theorem is proved in [3] for $\mu =$ Lebesgue measure. The proof, with easy modifications, works for doubling measures.

For functions in $L^1(\mu) + L^\infty(\mu)$ we get from (3.6) a similar inequality with no restriction on t.

We apply (3.6) to $fI_{Q(r)}$ where $Q(r)$ is the cube centered at 0 with side length r. We get

$$\frac{1}{t}\int_0^t (fI_{Q(r)})^{*,\mu}(s)\,ds - (fI_{Q(r)})^{*,\mu}(t) \leqslant C(\mu,d)(f^{\#,\mu}_{Q(r)})^{*,\mu}(t)$$
$$\leqslant C(\mu,d)(f^{\#,\mu})^{*,\mu}(t).$$

Clearly, as $r \nearrow \infty$, we have $(fI_{Q(r)})^{*,\mu}(t) \nearrow f^{*,\mu}(t)$ and so also

$$\lim_{r\to\infty} \frac{1}{t}\int_0^t (fI_{Q(r)})^{*,\mu}(s)\,ds = \frac{1}{t}\int_0^t f^{*,\mu}(s)\,ds$$

so that we have for all t,

$$f^{**,\mu}(t) - f^{*,\mu}(t) = \frac{1}{t}\int_0^t f^{*,\mu}(s)\,ds - f^{*,\mu}(t) \leqslant C(\mu,d)(f^{\#,\mu})^{*,\mu}(t) \tag{3.7}$$

provided $f \in L^1(\mu) + L^\infty(\mu)$.

Since $||f^{\#,\mu}||_\infty \backsim ||f||_{BMO(\mu)}$, (3.7) implies

$$\sup_{t>0}\left(\frac{1}{t}\int_0^t f^{*,\mu}(s)\,ds - f^{*,\mu}(t)\right) \leqslant C(\mu,d)||f||_{BMO(\mu)}. \tag{3.8}$$

Bennett, DeVore, and Sharpley showed in [3] that the class of functions which satisfy

$$\sup_{t>0}\left(\frac{1}{t}\int_0^t f^{*,\mu}(s)\,ds - f^{*,\mu}(t)\right) < \infty$$

is the rearrangement-invariant hull of BMO, which they called weak-L^∞. There is no established notation for weak-L^∞; we adopt Milman's notation [12] and write

$$||f||_{WL^\infty(\mu)} = \sup_{t>0}\left(\frac{1}{t}\int_0^t f^{*,\mu}(s)\,ds - f^{*,\mu}(t)\right).$$

We will derive several expressions for the norms in weak type classes, depending on the interpolation scales. The following two lemmas will be used to prove the equivalence of the different expressions.

LEMMA 3.2. *Let $\gamma > 1$ and $1 \leqslant r < \infty$. If $0 \leqslant h$ is a non-increasing function then*

$$\frac{\gamma-1}{\gamma r}\left[\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - h(t)\right] \leqslant \left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - \left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}}$$
$$\leqslant \left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - h(\gamma t)$$
$$\leqslant \gamma\left[\left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}} - h(\gamma t)\right].$$

Proof. Let us consider the first inequality. We have

$$\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - \left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}}$$
$$\geqslant \frac{1}{r}\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}-1}\left(\frac{1}{t}\int_0^t h^r - \frac{1}{\gamma t}\int_0^t h^r - \frac{1}{\gamma t}\int_t^{\gamma t} h^r\right)$$
$$= \frac{1}{r}\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}-1}\left(\frac{1-\gamma^{-1}}{t}\int_0^t h^r - \frac{1}{\gamma t}\int_t^{\gamma t} h^r\right)$$
$$= \frac{1-\gamma^{-1}}{r}\left[\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - \left(\frac{\frac{1}{(\gamma-1)t}\int_t^{\gamma t} h^r}{\frac{1}{t}\int_0^t h^r}\right)^{1-\frac{1}{r}}\left(\frac{1}{(\gamma-1)t}\int_t^{\gamma t} h^r\right)^{\frac{1}{r}}\right]$$
$$\geqslant \frac{1-\gamma^{-1}}{r}\left[\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - \left(\frac{1}{(\gamma-1)t}\int_t^{\gamma t} h^r\right)^{\frac{1}{r}}\right]$$
$$\geqslant \frac{1-\gamma^{-1}}{r}\left[\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - h(t)\right].$$

The second inequality is trivial. For the third inequality we observe first that from

$$\frac{1}{t(\gamma-1)}\int_t^{\gamma t} h \geqslant h(\gamma t),$$

which holds for $0 \leqslant h \searrow$ and $\gamma > 1$ follows

$$\frac{1}{t}\int_0^t h - h(\gamma t) \leqslant \gamma\left(\frac{1}{\gamma t}\int_0^{\gamma t} h - h(\gamma t)\right).$$

We apply this inequality below:

$$\begin{aligned}
\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - h(\gamma t) &\leqslant \frac{1}{r}\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}-1}\left[\frac{1}{t}\int_0^t h^r - h^r(\gamma t)\right] \\
&\leqslant \frac{\gamma}{r}\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}-1}\left[\frac{1}{\gamma t}\int_0^{\gamma t} h^r - h^r(\gamma t)\right] \\
&= \frac{\gamma}{r}\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}-1}\left[\left(\left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}}\right)^r - h^r(\gamma t)\right] \\
&\leqslant \gamma\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}-1}\left(\left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}}\right)^{r-1} \\
&\times \left[\left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}} - h(\gamma t)\right] \\
&\leqslant \gamma\left[\left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}} - h(\gamma t)\right]. \quad \blacksquare
\end{aligned}$$

LEMMA 3.3. *If $0 \leqslant h \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is a non-increasing function and $\gamma > 1$ then*

$$\sup_{t>0}\, [h(t) - h(\gamma t)] \backsim \sup_{t>0}\left(\frac{1}{t}\int_0^t h - h(t)\right).$$

For $r \geqslant 1$, we also have

$$\begin{aligned}
\sup_{t>0}\left(\frac{1}{t}\int_0^t h - h(t)\right) &\backsim \sup_{t>0}\left(\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - h(t)\right) \\
&\backsim \sup_{t>0}\left(\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}} - \left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}}\right)
\end{aligned}$$

with the constants of equivalence depending on γ and r only.

Proof. The first equivalence was proved in [14]. The equivalence

$$\sup_{t>0}\left(\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}}-h(t)\right)\backsim\sup_{t>0}\left(\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}}-\left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}}\right)$$

follows from Lemma 3.2. Since

$$\begin{aligned}\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}}-\left(\frac{1}{\gamma t}\int_0^{\gamma t} h^r\right)^{\frac{1}{r}}&=\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}}-\left(\frac{1}{t}\int_0^t h(\gamma s)^r\,ds\right)^{\frac{1}{r}}\\&\leqslant\left(\frac{1}{t}\int_0^t (h(s)-h(\gamma s))^r\,ds\right)^{\frac{1}{r}}\\&\leqslant\sup_{s>0}\,[h(s)-h(\gamma s)]\end{aligned}$$

and since

$$\frac{1}{t}\int_0^t h-h(t)\leqslant\left(\frac{1}{t}\int_0^t h^r\right)^{\frac{1}{r}}-h(t)$$

the lemma is proved. ∎

We interpret (3.8) in terms of $W_K[L^\infty(\mu),L(p,\infty)(\mu);1,\gamma,1,g]$ for $g(t,f)=f^{*,\mu}(t^{-p})$.

LEMMA 3.4. *If* $p>0,\gamma>1$, *and*

$$g(t,f)=f^{*,\mu}(t^{-p}),$$

then $W_K[L^\infty(\mu),L(p,\infty)(\mu);1,\gamma,1,g]$ *is well defined and*

$$||f||_{W_K[L^\infty(\mu),L(p,\infty)(\mu);1,\gamma,1,g]}\backsim||f||_{WL^\infty(\mu)}. \tag{3.9}$$

Proof. We need to show that $\hat{g}\backsim K(\cdot,\cdot;L^\infty(\mu),\ L(p,\infty)(\mu))$. From Holmstedt's formula, see [5], follows

$$K(t,f;L(p,\infty)(\mu),L^\infty(\mu))\backsim\sup_{0<s<t}\,sf^{*,\mu}(s^p).$$

Therefore

$$\begin{aligned}
&K(t,f;L^{\infty}(\mu),L(p,\infty)(\mu))\\
&\quad = tK\left(\frac{1}{t},f;L(p,\infty)(\mu),L^{\infty}(\mu)\right)\backsim t\sup_{0<s<\frac{1}{t}} sf^{*,\mu}(s^p)\\
&\quad = t\sup_{0<t<s}\frac{1}{s}f^{*,\mu}\left(\frac{1}{s^p}\right) = t\sup_{0<t<s}\frac{g(s,f)}{s}.
\end{aligned}$$

Let us see that

$$t\sup_{0<t<s}\frac{g(s,f)}{s}\backsim\hat{g}(t,f). \tag{3.10}$$

Clearly, from

$$t\sup_{0<t<s}\frac{g(s,f)}{s}\leqslant CK(t,f;L^{\infty}(\mu),L(p,\infty)(\mu))$$

follows

$$g(t,f)\leqslant CK(t,f;L^{\infty}(\mu),L(p,\infty)(\mu))$$

and since K is concave we have

$$\hat{g}(t,f)\leqslant CK(t,f;L^{\infty}(\mu),L(p,\infty)(\mu)).$$

Conversely,

$$K(t,f;L^{\infty}(\mu),L(p,\infty)(\mu))\leqslant Ct\sup_{0<t<s}\frac{g(s,f)}{s}$$

implies

$$K(t,f;L^{\infty}(\mu),L(p,\infty)(\mu))\leqslant Ct\sup_{0<t<s}\frac{\hat{g}(s,f)}{s}.$$

But since $\hat{g}$ is concave, we have that $\frac{\hat{g}(s,f)}{s}$ is a non-increasing function so that

$$t\sup_{0<t<s}\frac{\hat{g}(s,f)}{s}=\hat{g}(t,f)$$

and so (3.10) holds. This implies of course that $\hat{g} \backsim K(\cdot\,,\cdot\,;L^{\infty}(\mu), L(p,\infty)(\mu))$. Let us show (3.9). Since g is non-decreasing, we have

$$\begin{aligned} \|f\|_{W_K[L^{\infty}(\mu),L(p,\infty)(\mu);1,\gamma,1,g]} &= \sup_{t>0}[g(\gamma t)-g(t)]_+ = \sup_{t>0}[g(\gamma t)-g(t)] \\ &= \sup_{t>0}\left[f^{*,\mu}\left(\frac{1}{(\gamma t)^p}\right)-f^{*,\mu}\left(\frac{1}{t^p}\right)\right] \\ &= \sup_{t>0}\ [f^{*,\mu}(t)-f^{*,\mu}(\gamma^p t)] \end{aligned}$$

and by Lemma 3.3

$$\|f\|_{W_K[L^{\infty}(\mu),L(p,\infty)(\mu);1,\gamma,1,g]} \backsim \|f\|_{WL^{\infty}(\mu)}. \quad \blacksquare$$

Lemma 3.5. *Let* $p \geqslant 1,\ \gamma > 1$, *and* $g(t,f) = t(\int_0^{t^{-p}} (f^{*,\mu})^p)^{\frac{1}{p}}$. *Then the class* $W_K(L^{\infty}(\mu), L^p(\mu); 1, \gamma, 1, g)$ *is well defined and*

$$\|f\|_{W_K(L^{\infty}(\mu),L^p(\mu);1,\gamma,1,g)} \backsim \|f\|_{WL^{\infty}(\mu)}. \tag{3.11}$$

Proof. From Holmstedt's formula

$$K(t,f;L^{\infty}(\mu),L^p(\mu)) \backsim t\left(\int_0^{t^{-p}} (f^{*,\mu})^p\right)^{\frac{1}{p}} = g(t,f) \tag{3.12}$$

so that $W_K(L^{\infty}(\mu), L^p(\mu); 1, \gamma, 1, g)$ is well defined.

Moreover,

$$\begin{aligned} \|f\|_{W_K(L^{\infty}(\mu),L^p(\mu);1,\gamma,1,g)} &= \sup_{t>0}\ (g(\gamma t,f)-g(t,f)) \\ &= \sup_{t>0}\left(\gamma t\left(\int_0^{(\gamma t)^{-p}} (f^{*,\mu})^p\right)^{\frac{1}{p}} - t\left(\int_0^{t^{-p}} (f^{*,\mu})^p\right)^{\frac{1}{p}}\right) \\ &= \sup_{s>0}\left(\left(\frac{1}{s}\int_0^{s} (f^{*,\mu})^p\right)^{\frac{1}{p}} - \left(\frac{1}{\gamma^p s}\int_0^{\gamma^p s} (f^{*,\mu})^p\right)^{\frac{1}{p}}\right) \end{aligned}$$

and from Lemma 3.3 we have (3.11). $\blacksquare$

Theorem 3.6. *Let* $T: A_0 + A_1 \to L^{\infty}(\mu) + L(p,\infty)(\mu)$ *be a subadditive operator so that for* $p > 0$,

$$\|Ta\|_{WL^{\infty}(\mu)} \leqslant M_0\|a\|_{A_0},$$

$$\|Ta\|_{L(p,\infty)(\mu)} \leqslant M_1\|a\|_{A_1}.$$

Then for all $t > 0$ *and* $\gamma > 1$,

$$(Ta)^{*,\mu}(t) - (Ta)^{*,\mu}(\gamma t) \leqslant CM_0K\left(\frac{M_1}{M_0}t^{-\frac{1}{p}}, a; A_0, A_1\right), \tag{3.13}$$

where $C = C(\gamma, p)$.

Proof. Let $g(t,f) = f^{*,\mu}(t^{-p})$ so that, by (3.9),

$$||f||_{W_K[L^\infty(\mu), L(p,\infty)(\mu); 1, \gamma^{\frac{1}{p}}, 1, g]} \backsim ||f||_{WL^\infty(\mu)}.$$

To apply Theorem 2.8 we need to show first that (2.4) holds.

Let $f_1 \in L^\infty(\mu) + L(p,\infty)(\mu)$ and $f_2 \in L(p,\infty)(\mu)$ and let $\lambda = (\frac{\gamma}{\gamma-1})^{\frac{1}{p}}$. We have

$$\begin{aligned}
g(t, f_1 + f_2) &= (f_1 + f_2)^{*,\mu}\left(\frac{1}{t^p}\right) = (f_1 + f_2)^{*,\mu}\left(\frac{1}{\gamma t^p} + \frac{\gamma - 1}{\gamma t^p}\right) \\
&\leqslant f_1^{*,\mu}\left(\frac{1}{\gamma t^p}\right) + f_2^{*,\mu}\left(\frac{\gamma - 1}{\gamma t^p}\right) \\
&\leqslant g\left(\gamma^{\frac{1}{p}} t, f_1\right) + \left(\frac{\gamma}{\gamma - 1}\right)^{\frac{1}{p}} t ||f_2||_{L(p,\infty)(\mu)} \\
&= g(\gamma^{\frac{1}{p}} t, f_1) + \lambda t ||f_2||_{L(p,\infty)(\mu)}
\end{aligned}$$

so that (2.4) holds.

By Theorem 2.8, and since the K-functional is a non-decreasing function,

$$\begin{aligned}
g(\gamma^{\frac{1}{p}} t, Ta) - g(t, Ta) &\leqslant g^\wedge(t, Ta, \gamma^{\frac{1}{p}}, 1; L^\infty(\mu), L^{p,\infty}(\mu)) \\
&\leqslant CM_0K\left(\frac{M_1}{M_0} t, a; A_0, A_1\right) \\
&\leqslant CM_0K\left(\frac{M_1}{M_0} \gamma^{\frac{1}{p}} t, a; A_0, A_1\right)
\end{aligned}$$

so that

$$(Ta)^{*,\mu}(\gamma^{-1} t^{-p}) - (Ta)^{*,\mu}(t^{-p}) \leqslant CM_0K\left(\frac{M_1}{M_0} \gamma^{\frac{1}{p}} t, a; A_0, A_1\right)$$

which proves (3.13). ∎

COROLLARY 3.7. *Let μ and ν be doubling measure and let*

$$T : L^{\infty}(\nu) + L^{q}(\nu) \to L^{\infty}(\mu) + L(p,\infty)(\mu)$$

be a subadditive operator. Assume that for $p > 0$, $0<q<\infty$,

$$||Tf||_{BMO(\mu)} \leqslant M_0 ||f||_{L^{\infty}(\nu)},$$

$$||Tf||_{L(p,\infty)(\mu)} \leqslant M_1 ||f||_{L^{q}(\nu)}.$$

Then

$$(Tf)^{*,\mu}(t) - (Tf)^{*,\mu}(\gamma t) \leqslant C \mathscr{M}_{\nu}(|f|^{q})^{*,\nu}\left(t^{\frac{q}{p}}\right), \tag{3.14}$$

where $C = C(M_0, M_1, \gamma, \mu, \nu, p, q)$.

Proof. Since μ is a doubling measure,

$$||Tf||_{WL^{\infty}(\mu)} \leqslant C ||Tf||_{BMO(\mu)} \tag{3.15}$$

and we can apply the previous theorem. We do so with $A_0 = L^{\infty}(\nu)$ and $A_1 = L^{q}(\nu)$. Using (3.12), we get

$$(Tf)^{*,\mu}(t) - (Tf)^{*,\mu}(\gamma t) \leqslant C M_0 \left(\frac{1}{\left(\frac{M_0 t^{\frac{1}{p}}}{M_1}\right)^{q}} \int_0^{\left(\frac{M_0 t^{\frac{1}{p}}}{M_1}\right)^{q}} (f^{*,\nu})^{q} \right)^{\frac{1}{q}}.$$

Since the right-hand side is a K-functional, we have

$$(Tf)^{*,\mu}(t) - (Tf)^{*,\mu}(\gamma t) \leqslant C \left(\frac{1}{t^{\frac{q}{p}}} \int_0^{t^{\frac{q}{p}}} (f^{*,\nu})^{q} \right)^{\frac{1}{q}} = [(|f|^{q})^{**,\nu}\left(t^{\frac{q}{p}}\right)]^{\frac{1}{q}}.$$

Since ν satisfies a doubling condition,

$$(|f|^{q})^{**,\nu}\left(t^{\frac{q}{p}}\right) \backsim \mathscr{M}_{\nu}(|f|^{q})^{*,\nu}\left(t^{\frac{q}{p}}\right).$$

This is Herz's Theorem, see for example, [4, Theorem 3.8, p. 122] where this is proved for $\nu =$ Lebesgue measure. The proof, with minor modifications, holds for doubling measures.

Thus we have shown (3.14). ∎

Remark 3.8. Observe that if $p = q = 1$, $\mu = v =$ Lebesgue measure, we get (3.2).

COROLLARY 3.9. *Let μ and v be doubling measure and let*

$$T : BMO(v) + L^q(v) \to BMO(\mu) + L(p, \infty)(\mu)$$

be a subadditive operator. Assume that for $p > 0$ and $0 < q < \infty$,

$$||Tf||_{BMO(\mu)} \leqslant M_0 ||f||_{BMO(v)},$$

$$||Tf||_{L(p,\infty)(\mu)} \leqslant M_1 ||f||_{L^q(v)}.$$

Then

$$(Tf)^{*,\mu}(t) - (Tf)^{*,\mu}(\gamma t) \leqslant C(f_q^{\#,v})^{*,v}\left(t^{\frac{q}{p}}\right), \tag{3.16}$$

where $C = C(M_0, M_1, \gamma, \mu, v, p, q)$.

Proof. As in the previous corollary, since μ is a doubling measure, we have (3.15) and we can apply Theorem 3.6. We do so with $A_0 = BMO(v)$ and $A_1 = L^q(v)$. From Corollary 3.5 and Remark 3.7 in [8]

$$K(t, f; BMO(v), L^q(v)) \backsim (f_q^{\#,v})^{*,v}(t^{-q})$$

so that we have (3.16). ∎

Remark 3.10. Observe that if $q = p = 1$ and $\mu = v =$ Lebesgue measure then

$$(Tf)^*(t) - (Tf)^*(\gamma t) \leqslant C(f^\#)^*(t)$$

and we have proved (3.3).

THEOREM 3.11. *Let $T : A_0 + A_1 \to L^\infty(\mu) + L^p(\mu)$ be a subadditive operator so that for $p \geqslant 1$,*

$$||Ta||_{WL^\infty(\mu)} \leqslant M_0 ||a||_{A_0},$$

$$||Ta||_{L^p(\mu)} \leqslant M_1 ||a||_{A_1}.$$

Then for all $t > 0$ *and* $\gamma > 1$,

$$\left(\frac{1}{t}\int_0^t ((Ta)^{*,\mu})^p\right)^{\frac{1}{p}} - \left(\frac{1}{\gamma t}\int_0^{\gamma t} ((Ta)^{*,\mu})^p\right)^{\frac{1}{p}} \leqslant CM_0 K\left(\frac{M_1}{M_0} t^{-\frac{1}{p}}, a; A_0, A_1\right), \tag{3.17}$$

where $C = C(\gamma, p)$.

Proof. Let

$$g(t, f) = t\left(\int_0^{t^{-p}} (f^{*,\mu})^p\right)^{\frac{1}{p}}$$

so that by (3.11)

$$||Ta||_{W_K(L^\infty(\mu), L^p(\mu); 1, \gamma, 1, g)} \backsim ||Ta||_{WL^\infty(\mu)}.$$

Let us see that condition (2.4) holds. We have, see [6],

$$\left(\int_0^t ((f_1 + f_2)^{*,\mu})^p\right)^{\frac{1}{p}} \leqslant \left(\int_0^t (f_1^{*,\mu})^p\right)^{\frac{1}{p}} + \left(\int_0^t (f_2^{*,\mu})^p\right)^{\frac{1}{p}}$$

so that if $f_1 \in L^\infty(\mu) + L^p(\mu)$ and $f_2 \in L^p(\mu)$

$$\begin{aligned} g(t, f_1 + f_2) &= t\left(\int_0^{t^{-p}} ((f_1 + f_2)^{*,\mu})^p\right)^{\frac{1}{p}} \\ &\leqslant t\left(\int_0^{t^{-p}} (f_1^{*,\mu})^p\right)^{\frac{1}{p}} + t\left(\int_0^{t^{-p}} (f_2^{*,\mu})^p\right)^{\frac{1}{p}} \\ &\leqslant t\left(\int_0^{t^{-p}} (f_1^{*,\mu})^p\right)^{\frac{1}{p}} + t\left(\int_0^{\infty} (f_2^{*,\mu})^p\right)^{\frac{1}{p}} \\ &= g(t, f_1) + t||f_2||_{L^p(\mu)}. \end{aligned}$$

By Theorem 2.8,

$$g\left(\gamma^{\frac{1}{p}} t, Ta\right) - g(t, Ta) \leqslant CM_0 K\left(\frac{M_1}{M_0} t, a; A_0, A_1\right)$$

so that

$$\gamma^{\frac{1}{p}} t\left(\int_0^{\gamma^{-1}t^{-p}} ((Ta)^{*,\mu})^p\right)^{\frac{1}{p}} - t\left(\int_0^{t^{-p}} ((Ta)^{*,\mu})^p\right)^{\frac{1}{p}}$$
$$\leqslant CM_0 K\left(\frac{M_1}{M_0}t, a; A_0, A_1\right)$$

which implies (3.17). ∎

Corollary 3.12. *Applying Lemma* 3.2 *to* (3.17) *we get*

$$\left(\frac{1}{t}\int_0^t ((Ta)^{*,\mu})^p\right)^{\frac{1}{p}} - (Ta)^{*,\mu}(\gamma t) \leqslant CM_0 K\left(\frac{M_1}{M_0}t^{-\frac{1}{p}}, a; A_0, A_1\right). \tag{3.18}$$

Remark 3.13. The hypotheses of Theorems 3.6 and 3.11 imply the same interpolation result: for $0<\theta<1$ we have

$$||Ta||_{L^{p_\theta}(\mu)} \leqslant C(\theta, p)||a||_{(A_0,A_1)_{\theta,p_\theta}},$$

where $p_\theta = \frac{p}{1-\theta}$. The rearrangement-function inequalities are more sensitive to the endpoint conditions. For $p \geqslant 1$, inequality (3.13) which we get from the weaker condition,

$$||Ta||_{L(p,\infty)(\mu)} \leqslant M_1||a||_{A_1},$$

in Theorem 3.6, is strictly weaker than inequality (3.18) which we get in Corollary 3.12.

To see that we take $A_0 = L^p$, $A_1 = L^\infty$, both spaces taken on $\mathbb{R}_+$ with Lebesgue measure, and denote by C_W and C_S two constants for which the inequalities are valid. Thus we want to show that

$$\left(\frac{1}{t}\int_0^t ((Tf)^*)^p\right)^{\frac{1}{p}} - (Tf)^*(\gamma t) \leqslant C_S M_1' \left(\frac{1}{t}\int_0^{\left(\frac{M_0'}{M_1'}\right)^p t} (f^*)^p\right)^{\frac{1}{p}} \tag{3.19}$$

is strictly stronger than

$$(Tf)^*(t) - (Tf)^*(\gamma t) \leqslant C_W M_1 \left(\frac{1}{t}\int_0^{\left(\frac{M_0}{M_1}\right)^p t} (f^*)^p\right)^{\frac{1}{p}}. \tag{3.20}$$

We take

$$Tf = \mathcal{M}_p f = \mathcal{M}^{\frac{1}{p}}(|f|^p),$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal function. The operator maps L^p to $L(p, \infty)$ and L^∞ to itself and so certainly to WL^∞. It follows that (3.20) holds. Let us see that (3.19) does not hold:

For every $\varepsilon > 0$ let

$$f_\varepsilon(t) = f_\varepsilon^*(t) = t^{-\frac{1}{p+\varepsilon}} I_{(0,1)}(t).$$

Thus, for $0 < t < 1$,

$$(\mathcal{M}_p f_\varepsilon)^*(t) = \mathcal{M}_p f_\varepsilon(t) = \left(\frac{1}{t}\int_0^t (f_\varepsilon^*)^p\right)^{\frac{1}{p}} = \left(\frac{p+\varepsilon}{\varepsilon}\right)^{\frac{1}{p}} t^{-\frac{1}{p+\varepsilon}}$$

so that

$$\left(\frac{1}{t}\int_0^t ((\mathcal{M}_p f_\varepsilon)^*)^p\right)^{\frac{1}{p}} = \left(\frac{p+\varepsilon}{\varepsilon}\right)^{\frac{2}{p}} t^{-\frac{1}{p+\varepsilon}}.$$

Inequality (3.19) would imply for all $\varepsilon > 0$

$$\begin{aligned}\left(\frac{p+\varepsilon}{\varepsilon}\right)^{\frac{2}{p}} t^{-\frac{1}{p+\varepsilon}} &\leqslant C_W M_1 \left(\frac{1}{t}\int_0^{\left(\frac{M_0}{M_1}\right)^p t} (f_\varepsilon^*)^p\right)^{\frac{1}{p}} + (M_p f_\varepsilon)^*(\gamma t) \\ &= \left(\frac{p+\varepsilon}{\varepsilon}\right)^{\frac{1}{p}} \left(C_W M_1 \left(\frac{M_0}{M_1} t\right)^{-\frac{1}{p+\varepsilon}} + (\gamma t)^{-\frac{1}{p+\varepsilon}}\right)\end{aligned}$$

and so does not hold.

Let us consider a version of Theorem 3.11 for $0 < p < 1$.

THEOREM 3.14. *Let* $T : A_0 + A_1 \to L^\infty(\mu) + L^p(\mu)$ *be a subadditive operator so that for* $0 < p < 1$,

$$|||Ta|^p||_{WL^\infty(\mu)}^{\frac{1}{p}} \leqslant M_0 ||a||_{A_0},$$

$$||Ta||_{L^p(\mu)} \leqslant M_1 ||a||_{A_1}.$$

Then for all $t > 0$ and $\gamma > 1$,

$$\left(\frac{1}{t}\int_0^t ((Ta)^{*,\mu})^p - \frac{1}{\gamma t}\int_0^{\gamma t} ((Ta)^{*,\mu})^p\right)^{\frac{1}{p}} \leqslant CM_0 K\left(\frac{M_1}{M_0}t^{-\frac{1}{p}}, a; A_0, A_1\right). \tag{3.21}$$

Proof. We take

$$g(t,f) = t\left(\int_0^{t^{-p}} (f^{*,\mu})^p\right)^{\frac{1}{p}}.$$

Thus

$$\begin{aligned}
&||f||_{W_K(L^\infty(\mu),L^p(\mu);1,\gamma,p,g)} \\
&= \sup_{t>0} \left(g^p(\gamma t,f) - g^p(t,f)\right)^{\frac{1}{p}} \\
&= \sup_{t>0}\left(\gamma^p t^p\left(\int_0^{(\gamma t)^{-p}} (f^{*,\mu})^p\right) - t^p\left(\int_0^{t^{-p}} (f^{*,\mu})^p\right)\right)^{\frac{1}{p}} \\
&= \sup_{s>0}\left(\frac{1}{s}\int_0^s (f^{*,\mu})^p - \frac{1}{\gamma^p s}\int_0^{\gamma^p s} (f^{*,\mu})^p\right)^{\frac{1}{p}}
\end{aligned}$$

and, from Lemma 3.3 with $h = (f^{*,\mu})^p$ and $r = 1$, we have

$$\begin{aligned}
||f||_{W_K(L^\infty(\mu),L(p,\infty)(\mu);1,\gamma,p,g)} &\backsim \sup_{t>0}\left(\frac{1}{t}\int_0^t (f^{*,\mu})^p - (f^{*,\mu}(t))^p\right)^{\frac{1}{p}} \\
&= |||f|^p||^{\frac{1}{p}}_{WL^\infty(\mu)}.
\end{aligned}$$

Let us see that condition (2.4) with $r = p$ holds. If $f_1 \in L^\infty(\mu) + L^p(\mu)$ and $f_2 \in L^p(\mu)$ then

$$\begin{aligned}
g^p(t,f_1+f_2) &= t^p\int_0^{t^{-p}} ((f_1+f_2)^{*,\mu}(s))^p\,ds \\
&\leqslant t^p\int_0^{t^{-p}} (f_1^{*,\mu}(\gamma^{-1}s) + f_2^{*,\mu}((1-\gamma^{-1})s))^p\,ds \\
&\leqslant t^p\int_0^{t^{-p}} (f_1^{*,\mu}(\gamma^{-1}s))^p\,ds + t^p\int_0^{t^{-p}} (f_2^{*,\mu}((1-\gamma^{-1})s))^p\,ds
\end{aligned}$$

$$\leqslant \gamma t^p \int_0^{\gamma^{-1}t^{-p}} (f_1^{*,\mu}(s))^p \, ds + \frac{\gamma}{\gamma-1} t^p \int_0^{\infty} (f_2^{*,\mu}(s))^p \, ds$$
$$= g^p(\gamma^{\frac{1}{p}} t, f_1) + \frac{\gamma}{\gamma-1} t^p ||f_2||^p_{L^p(\mu)}.$$

By Theorem 2.8

$$\left(g^p\left(\gamma^{\frac{1}{p}} t, Ta\right) - g^p(t, Ta)\right)^{\frac{1}{p}} \leqslant CM_0 K\left(\frac{M_1}{M_0} t, a; A_0, A_1\right)$$

so that

$$\left(\gamma t^p \int_0^{\gamma^{-1}t^{-p}} ((Ta)^{*,\mu})^p - t^p \int_0^{t^{-p}} ((Ta)^{*,\mu})^p\right)^{\frac{1}{p}}$$
$$\leqslant CM_0 K\left(\frac{M_1}{M_0} t, a; A_0, A_1\right)$$

which implies (3.21). ∎

Corollary 3.15. *Applying Lemma* 3.2 *with* $h = (Tf)^{*,\mu}$ *and* $r = 1$, *to* (3.21) *we get*

$$\left(\frac{1}{t}\int_0^t ((Ta)^{*,\mu})^p - ((Ta)^{*,\mu})^p(\gamma t)\right)^{\frac{1}{p}} \leqslant CM_0 K\left(\frac{M_1}{M_0} t^{-\frac{1}{p}}, a; A_0, A_1\right).$$

4. CALCULATION OF THE E-FUNCTIONAL FOR SOME WEAK-TYPE CLASSES

Definition 4.1. Let (A_0, A_1) be an interpolation couple of quasi-Banach groups. We define

$$E(t, a; A_0, A_1) = \inf\{||a - a_0||_{A_1} : ||a_0||_{A_0} \leqslant t\}.$$

This definition is consistent with that of [5]. To state the results in a form which is consistent with the results of Section 2 we will work with $E(t, a; A_1, A_0)$ and in the sequel we denote

$$E(t, a) = E(t, a; A_1, A_0).$$

Definition 4.2. Given $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ we define the greatest non-increasing minorant of h:

$$\overset{\searrow}{h}(t) = \inf_{s \leqslant t} h(s).$$

DEFINITION 4.3 (Sagher and Shvartsman [14]). Let $h:\mathbb{R}_+ \times (A_0 + A_1) \mapsto \mathbb{R}_+$. Assume that there exists $\beta \geqslant 1$ so that for all $a \in A_0 + A_1$ and all $t > 0$

$$\frac{1}{\beta} \overset{\searrow}{h}(\beta t, a) \leqslant E(t, a) \leqslant \beta \overset{\searrow}{h}\left(\frac{t}{\beta}, a\right). \tag{4.1}$$

We define for $0 < r < \infty$ and $\gamma > 1$

$$||a||_{W_E(A_0,A_1;\varepsilon,\gamma,r,h)} = \sup_{t>0}\left[h^r\left(\frac{t}{\gamma}, a\right) - \varepsilon^r h^r\left(\frac{t}{\varepsilon}, a\right)\right]_+^{\frac{1}{r}}$$

and

$$W_E(A_0, A_1; \varepsilon, \gamma, r, h) = \{a \in A_0 + A_1 : ||a||_{W_E(A_0,A_1;\varepsilon,\gamma,r,h)} < \infty\}.$$

In this paper we consider only the case $\varepsilon = 1$.
We denote $W_E(A_0, A_1; 1, \gamma, r, E)$ by $W_E(A_0, A_1; 1, \gamma, r)$.
We reserve the letter h in this section for functions which satisfy (4.1).

DEFINITION 4.4.

$$h^\flat(t, a) = h^\flat(t, a, \gamma, r; A_0, A_1) = \sup_{s \geqslant t}\left(h^r\left(\frac{s}{\gamma}, a\right) - h^r(s, a)\right)_+^{\frac{1}{r}}. \tag{4.2}$$

The next theorem is the analog, for the E-functional, of Theorem 2.3.

THEOREM 4.5. *Assume that there exists an integer*, $m \geqslant 0$, *so that for all* $a \in A_0 + A_1$ *and all* $a' \in A_1$ *such that* $||a'||_{A_1} \leqslant t$

$$h(\gamma^m t, a + a') \leqslant h(t, a). \tag{4.3}$$

Then

$$h^\flat(\gamma^{m+1} t, a) \leqslant cE(t, a; A_1, W_E(A_0, A_1; 1, \gamma, r, h)),$$

where $c = (2m + 1)^{\frac{1}{r}}$.

Proof. We denote

$$\tilde{A}_0 = W_E(A_0, A_1; 1, \gamma, r, h).$$

Let $\eta > 0$ and $a_1 \in A_1$ be such that $||a_1||_{A_1} \leqslant t$, $a - a_1 \in \tilde{A}_0$, and so that

$$||a - a_1||_{\tilde{A}_0} \leqslant E(t, a; A_1, \tilde{A}_0) + \eta.$$

For all $s \geqslant \gamma^{m+1} t$ we have

$$h^r\left(\frac{s}{\gamma}, a\right) - h^r(s,a) = h^r\left(\frac{s}{\gamma}, a\right) - h^r(\gamma^{-m-1}s, a-a_1) + h^r(\gamma^{-m-1}s, a-a_1) - h^r(\gamma^m s, a-a_1) + h^r(\gamma^m s, a-a_1) - h^r(s,a).$$

Since $||a_1||_{A_1} \leqslant t \leqslant \gamma^{-m-1}s$, by (4.3) we have $h(\gamma^m s, a-a_1) \leqslant h(s,a)$. We also have

$$h\left(\frac{s}{\gamma}, a\right) \leqslant h(\gamma^{-m-1}s, a-a_1)$$

so that

$$\begin{aligned} h^r\left(\frac{s}{\gamma}, a\right) - h^r(s,a) &\leqslant h^r(\gamma^{-m-1}s, a-a_1) - h^r(\gamma^m s, a-a_1) \\ &= \sum_{j=0}^{2m} (h^r(\gamma^{j-m-1}s, a-a_1) - h^r(\gamma^{j-m}s, a-a_1)) \\ &\leqslant (2m+1)||a-a_1||^r_{\tilde{A}_0} \\ &\leqslant (2m+1)(E(t,a;A_1,\tilde{A}_0)+\eta)^r. \end{aligned}$$

Thus

$$h^\flat(\gamma^{m+1}t, a) \leqslant (2m+1)^{\frac{1}{r}} E(t,a;A_1,\tilde{A}_0). \quad \blacksquare$$

The next theorem shows that Theorem 4.5 is, in a sense, best possible. If we take $h = E(\cdot\,,\cdot\,;A_1,A_0)$ we have:

THEOREM 4.6. *Let c_1 be the constant in the quasi-triangle inequality for A_1 and let m be such that $2c_1 \leqslant \gamma^m$. Then for all $t>0$ and $a \in A_0 + A_1$ we have*

$$\frac{1}{c} E^\flat(\gamma^{m+1}t, a) \leqslant E(t,a;A_1, W_E(A_0,A_1;1,\gamma,r)) \leqslant cE^\flat(\gamma t, a), \tag{4.4}$$

where $c = (2m+1)^{\frac{1}{r}}$.

Proof. We denote

$$\tilde{A}_0 = W_E(A_0,A_1;1,\gamma,r).$$

Let us prove that (4.3) holds with m so that $2c_1 \leqslant \gamma^m$.

Let $a_1 \in A_1$ be such that $||a_1||_{A_1} \leqslant t,\ a - a_1 \in A_0$, and

$$||a - a_1||_{A_0} \leqslant E(t, a) + \eta.$$

Let $a' \in A_1$ be such that $||a'||_{A_1} \leqslant t$. Then $||a_1 + a'||_{A_1} \leqslant 2c_1 t \leqslant \gamma^m t$ so that

$$E(\gamma^m t, a + a') \leqslant ||(a + a') - (a_1 + a')||_{A_0} \leqslant E(t, a) + \eta$$

and (4.3) holds.

From the previous theorem

$$\frac{1}{(2m+1)^{\frac{1}{r}}} E^{\flat}(\gamma^{m+1} t, a) \leqslant E(t, a; A_1, \tilde{A}_0).$$

Conversely, given $\eta > 0$ we have $a_1 \in A_1$ so that $||a_1||_{A_1} \leqslant t,\ a - a_1 \in A_0$ and so that $||a - a_1||_{A_0} \leqslant E(t, a) + \eta$. We will show that

$$\begin{aligned} E(t, a; A_1, \tilde{A}_0) &\leqslant ||a - a_1||_{\tilde{A}_0} \\ &= \sup_{s>0} \left(E^r\left(\frac{s}{\gamma}, a - a_1\right) - E^r(s, a - a_1) \right)^{\frac{1}{r}} \\ &\leqslant (2m+1)^{\frac{1}{r}} E^{\flat}(\gamma t, a) + \eta. \end{aligned} \tag{4.5}$$

We consider first $s \geqslant \gamma^{m+1} t$:

$$\begin{aligned} E^r\left(\frac{s}{\gamma}, a - a_1\right) - E^r(s, a - a_1) = E^r\left(\frac{s}{\gamma}, a - a_1\right) &- E^r(\gamma^{-m-1} s, a) \\ + E^r(\gamma^{-m-1} s, a) &- E^r(\gamma^m s, a) \\ + E^r(\gamma^m s, a) &- E^r(s, a - a_1). \end{aligned}$$

But by (4.3)

$$E^r(\gamma^m s, a) - E^r(s, a - a_1) \leqslant 0$$

and

$$E^r\left(\frac{s}{\gamma}, a - a_1\right) - E^r(\gamma^{-m-1} s, a) \leqslant 0$$

and since $||a_1||_{A_1} \leqslant t \leqslant \gamma^{-m-1}s$,

$$
\begin{aligned}
E^r\left(\frac{s}{\gamma}, a - a_1\right) - E^r(s, a - a_1) &\leqslant E^r(\gamma^{-m-1}s, a) - E^r(\gamma^m s, a) \\
&= \sum_{j=0}^{2m} (E^r(\gamma^{j-m-1}s, a) - E^r(\gamma^{j-m}s, a)) \\
&\leqslant (2m+1)[E^\flat(\gamma t, a)]^r.
\end{aligned}
$$

If $0 < s \leqslant \gamma^{m+1}t$ then

$$
\begin{aligned}
E^r\left(\frac{s}{\gamma}, a - a_1\right) - E^r(s, a - a_1) &\leqslant ||a - a_1||_{A_0}^r - E^r(\gamma^{m+1}t, a - a_1) \\
&\leqslant E^r(t, a) + \eta - E^r(\gamma^{m+1}t, a - a_1) \\
&= E^r(t, a) + \eta - E^r(\gamma^{2m+1}t, a) \\
&\quad + E^r(\gamma^{2m+1}t, a) - E^r(\gamma^{m+1}t, a - a_1).
\end{aligned}
$$

But by (4.3)

$$E^r(\gamma^{2m+1}t, a) - E^r(\gamma^{m+1}t, a - a_1) \leqslant 0.$$

Also,

$$
\begin{aligned}
E^r(t, a) - E^r(\gamma^{2m+1}t, a) &= \sum_{j=0}^{2m} (E^r(\gamma^j t, a) - E^r(\gamma^{j+1}t, a)) \\
&\leqslant (2m+1)(E^\flat)^r(\gamma t, a).
\end{aligned}
$$

Thus for $0 < s \leqslant \gamma^{m+1}t$ also

$$E^r\left(\frac{s}{\gamma}, a - a_1\right) - E^r(s, a - a_1) \leqslant (2m+1)((E^\flat)^r(\gamma t, a) + \eta)$$

and so

$$E(t, a; A_1, \tilde{A}_0) \leqslant (2m+1)^{\frac{1}{r}}((E^\flat)^r(\gamma t, a) + \eta)^{\frac{1}{r}}$$

proving (4.5). ∎

Remark 4.7. If $||\cdot||_{A_1}$ is a non-Archimedian metric, i.e.,

$$||a_1 + a_2||_{A_1} \leqslant \max\{||a_1||_{A_1}; ||a_2||_{A_1}\}$$

then it is easy to see that $m = 0$ and hence

$$E^{\flat}(\gamma t, a) = E(t, a; A_1, W_E(A_0, A_1; 1, \gamma, r)).$$

5. APPLICATIONS OF THE E-FUNCTIONAL

We recall:

DEFINITION 5.1 (Peetre and Sparr [13]).

$$||f||_{L^0(\mu)} = \mu\{|f| > 0\}$$

and

$$L^0(\mu) = \{f : ||f||_{L^0(\mu)} < \infty\}.$$

LEMMA 5.2. *On any σ-finite measure space (Ω, Σ, μ),*

$$||f||_{W_E(L^\infty(\mu), L^0(\mu); 1, \gamma, 1)} \backsim ||f||_{WL^\infty(\mu)}.$$

Proof. Since

$$E(t, f; L^0(\mu), L^\infty(\mu)) = f^{*,\mu}(t) \tag{5.1}$$

see [13], we have by Lemma 3.3

$$\begin{aligned} &||f||_{W_E(L^\infty(\mu), L^0(\mu); 1, \gamma, 1)} \\ &\quad = \sup_{t>0} \left(E(t, f; L^0(\mu), L^\infty(\mu)) - E(\gamma t, f; L^0(\mu), L^\infty(\mu))\right) \\ &\quad = \sup_{t>0} \left(f^{*,\mu}(t) - f^{*,\mu}(\gamma t)\right) \backsim \sup_{t>0} \left(f^{**,\mu}(t) - f^{*,\mu}(t)\right) = ||f||_{WL^\infty(\mu)}. \quad \blacksquare \end{aligned}$$

For the rest of the paper, μ stands for a Borel measure on $\mathbb{R}^d$.
We define the John–Stromberg maximal function.

DEFINITION 5.3 (John [9] and Strömberg [16]). Given a measurable function, f, on $\mathbb{R}^d$, we define

$$\mathcal{M}_s^{\#,\mu} f(x) = \sup_{Q \ni x} \inf_c \, (f - c)^{*,\mu_Q}(s).$$

Again if $\mu =$ Lebesgue measure we omit mention of the measure. If we need to mention Lebesgue measure, we denote it by λ. We recall, [9]: $||f||_{BMO} \backsim ||\mathcal{M}_s^{\#} f||_{L^\infty}$ for all $s < \frac{1}{2}$. The constants of equivalence depend on

s, d, of course. A more delicate result, see [16], is

$$\left\|\sup_{Q\ni x}\ \inf_{c}\ (f-c)^{*,\lambda_Q}\left(\frac{1}{2}-\right)\right\|_{L^\infty} \backsim \|f\|_{BMO}.$$

If μ is a doubling measure then there exists $s_1 = s_1(d,\mu) > 0$ so that for all $0 < s < s_1$ we have, see [17]:

$$\|f\|_{BMO(\mu)} \backsim \|\mathscr{M}_s^{\#,\mu} f\|_{L^\infty(\mu)}.$$

A more precise result holds. Let $0 < p < \infty$. By Chebycheff's inequality we have for every cube, Q, constant, c, and any $0 < s \leqslant 1$

$$\left(\int |f-c|^p\, d\mu_Q\right)^{\frac{1}{p}} = \left(\int_0^1 (|f-c|^{*,\mu_Q}(t))^p\, dt\right)^{\frac{1}{p}} \geqslant |f-c|^{*,\mu_Q}(s) s^{\frac{1}{p}}$$

so that

$$\begin{aligned}\mathscr{M}_s^{\#,\mu} f(x) &= \sup_{Q\ni x}\ \inf_{c}\ (f-c)^{*,\mu_Q}(s)\\ &\leqslant s^{-\frac{1}{p}} \sup_{Q\ni x}\ \inf_{c}\left(\int |f-c|^p\, d\mu_Q\right)^{\frac{1}{p}} = s^{-\frac{1}{p}} f_p^{\#,\mu}(x). \qquad (5.2)\end{aligned}$$

Recall the definition of $\mathscr{M}_\mu$, see (3.4), and define

$$\mathscr{M}_{p,\mu} = [\mathscr{M}_\mu(|f|^p)]^{\frac{1}{p}}.$$

Let μ be a doubling measure. Then there exists a constant, $C = C(s,p,\mu,d)$ so that for $f \in L^p(\mu) + BMO(\mu)$ and $0 < s < s_1(\mu,d)$,

$$\frac{1}{C}\mathscr{M}_{p,\mu}\mathscr{M}_s^{\#,\mu} f(x) \leqslant f_p^{\#,\mu}(x) \leqslant C\mathscr{M}_{p,\mu}\mathscr{M}_s^{\#,\mu} f(x), \qquad (5.3)$$

where $f_p^{\#,\mu}$ is the Fefferman–Stein maximal operator, defined in (3.5). The proof of this inequality for Lebesgue measure in [8] holds in this context and (5.3) shows that taking $\mathscr{M}_{p,\mu}$ of $\mathscr{M}_s^{\#,\mu} f(x)$ in (5.2) transforms the inequality to an equivalence.

THEOREM 5.4. *If μ is a doubling measure on $\mathbb{R}^d$ and $f \in L^0(\mu) + L^\infty(\mu)$ then for all $0 < s < s_1(\mu,d)$ we have*

$$f^{*,\mu}(t) - f^{*,\mu}(\gamma t) \leqslant C(\mathscr{M}_s^{\#,\mu})^{*,\mu}\left(\frac{t}{C}\right), \qquad (5.4)$$

where $C = C(\mu,d,\gamma,s)$.

Proof. By (4.4) with $r = 1$ we have for m so that $\gamma^m \geqslant 2$,

$$E^\flat(\gamma^{m+1}t, f) \leqslant (2m+1)E(t, f; L^0(\mu), W_E(L^\infty(\mu), L^0(\mu); 1, \gamma, 1)).$$

By (5.1) and (4.2) it follows that

$$\begin{aligned} &f^{*,\mu}(\gamma^m t) - f^{*,\mu}(\gamma^{m+1}t) \\ &\quad \leqslant (2m+1)E(t, f; L^0(\mu), W_E(L^\infty(\mu), L^0(\mu); 1, \gamma, 1)). \end{aligned}$$

From Lemma 5.2,

$$||f||_{W_E(L^\infty(\mu), L^0(\mu); 1, \gamma, 1)} \backsim ||f||_{WL^\infty(\mu)}$$

and since, see (3.8),

$$||f||_{WL^\infty(\mu)} \leqslant C(\mu, d)||f||_{BMO(\mu)}$$

we have

$$f^{*,\mu}(\gamma^m t) - f^{*,\mu}(\gamma^{m+1}t) \leqslant C(\mu, d, \gamma)E(t, f; L^0(\mu), BMO(\mu)).$$

The E-functional between $L^0(\mu)$ and $BMO(\mu)$, where μ is a doubling measure, was calculated in [8]:

$$C_1(\mathscr{M}_s^{\#,\mu}f)^{*,\mu}(C_2 t) \leqslant E(t, f; L^0(\mu), BMO(\mu)) \leqslant C_3(\mathscr{M}_s^{\#,\mu}f)^{*,\mu}(C_4 t)$$

provided $0 < s < s_1(\mu, d)$ proving (5.4). ∎

Lerner [10] proved a related inequality. To state Lerner's inequality we need to recall the definition of A_∞ weights.

Definition 5.5. A positive function, $\omega \in L^1_{\text{loc}}$, is said to be an A_∞ weight if there exist constants $c_\omega, \delta > 0$ so that for all cubes Q and all measurable $E \subseteq Q$ we have

$$\frac{\int_E \omega}{\int_Q \omega} \leqslant C\left(\frac{\lambda(E)}{\lambda(Q)}\right)^\delta,$$

where λ is Lebesgue measure.

We write ω also for the measure $\omega(x)\,dx$. Thus we write $\omega(E)$ for $\int_E \omega$, and $\omega_Q(E) = \frac{\omega(E\cap Q)}{\omega(Q)}$. The A_∞ condition is therefore

$$\omega_Q(E) \leqslant c_\omega \lambda_Q^\delta(E).$$

It is well known that if $\omega \in A_\infty$ then ω is a doubling measure; also $L^\infty(\lambda) = L^\infty(\omega)$.

Lerner's theorem states: if $\omega \in A_\infty$ then there exists $s_0 = s_0(\omega) > 0$ so that for all $s \in (0, s_0)$

$$f^{*,\omega}(t) - f^{*,\omega}(2t) \leqslant 2(\mathcal{M}_s^{\#} f)^{*,\omega}(2t). \tag{5.5}$$

With the added hypothesis on the measure, (5.5) has the sharp-function, $\mathcal{M}_s^{\#} f$, i.e., taken with respect to Lebesgue measure, instead of with respect to the measure, $\mu = \omega\, dx$ as in (5.4).

Let us see how we can get (5.5) (with larger constants) from (5.4). From the definition of A_∞ follows that if f is a measurable function then for every cube, Q,

$$f^{*,\omega_Q}(t) \leqslant C(\omega) f^{*,\lambda_Q}\left(t^{\frac{1}{\delta}}\right).$$

Therefore,

$$\sup_Q \inf_c |f - c|^{*,\omega_Q}(t) \leqslant C(\omega) \sup_Q \inf_c |f - c|^{*,\lambda_Q}\left(t^{\frac{1}{\delta}}\right)$$

so that

$$\mathcal{M}_{s^\delta}^{\#,\omega} f(x) \leqslant C(\omega) \mathcal{M}_s^{\#} f(x)$$

see Remark 2.11 in [8]. From Theorem 5.4 it follows that for all $s \in (0, s_1^\delta)$

$$f^{*,\omega}(t) - f^{*,\omega}(\gamma t) \leqslant C(\mathcal{M}_s^{\#} f)^{*,\omega}\left(\frac{t}{C}\right) \tag{5.6}$$

and we have proved (5.5).

Bagby and Kurtz [1] proved that if $\omega \in A_\infty$ then for every $f \in L^1_{\text{loc}}(\omega)$ the following inequality holds:

$$f^{*,\omega}(t) \leqslant C(f^{\#})^{*,\omega}(2t) + f^{*,\omega}(2t). \tag{5.7}$$

The term $(f^{\#})^{*,\omega}$ is the non-increasing rearrangement of $f^{\#}$ with respect to the measure ω. The function $f^{\#}$ is however taken with respect to Lebesgue measure. The conclusion is, by (5.2), weaker than (5.6), and so the question is whether the hypotheses are the same. Indeed, in addition to the condition $f \in L^1_{\text{loc}}(\omega)$ there is an implicit assumption in [1], that $f^{*,\omega}$ exists. This together with $f \in L^1_{\text{loc}}(\omega)$ is equivalent to $f \in L^1(\omega) + L^\infty$ and so (5.6) implies (5.7).

It was also observed in [10] that (5.6) implies (3.1) with different constants. Using Lerner's ideas we can show a somewhat stronger theorem. We have shown

$$(Tf)^{*,\omega}(t) - (Tf)^{*,\omega}(\gamma t) \leqslant C(\mathcal{M}_s^{\#}(Tf))^{*,\omega}\left(\frac{t}{C}\right).$$

By Theorem 4.6 in [8], if T is Calderón–Zygmund operator which satisfies a certain continuity condition, A_{Φ}, then for all sufficiently small s,

$$\mathcal{M}_s^{\#}(Tf)(x) \leqslant C(\mathcal{M}f)(x)$$

so that

$$(Tf)^{*,\omega}(t) - (Tf)^{*,\omega}(\gamma t) \leqslant C(\mathcal{M}f)^{*,\omega}\left(\frac{t}{C}\right),$$

which is (3.1) with different constants.

If T is Calderón–Zygmund operator which satisfies a stronger continuity condition, see Theorem 4.7 in [8], then

$$\mathcal{M}_s^{\#}(Tf)(x) \leqslant C(f^{\#})(x),$$

which implies

$$(Tf)^{*,\omega}(t) - (Tf)^{*,\omega}(\gamma t) \leqslant C(f^{\#})^{*,\omega}\left(\frac{t}{C}\right).$$

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